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REVIEW

Stability of Fluid Motions. Volumes I and II. By DANIEL D. JOSEPH. Springer, 1976. 282 pp. (Vol. I), 274 pp. (Vol. II). £24.50 each.

Twenty years ago it could reasonably be claimed that a nonlinear theory of stability hardly existed. A few papers containing general results and discussions had been written and some intriguing suggestions made but essentially it was not exciting wide interest and progress was slow. Thus in Chandrasekhar's classical account *Hydrodynamic and Hydromagnetic Stability*, published in 1961, only 8 pages out of 652 were allotted to nonlinear aspects (and, significantly, these were mainly concerned with Stuart's seminal ideas). Nevertheless it was clearly recognized that nonlinearity plays a crucial role in the evolution of fluid motions from one type to another and in the transitional processes which convert a laminar flow into a turbulent one. This state of affairs was partly due to a lingering reluctance to accept the usefulness of even the linear theory in the shear flows of dominating interest at that time, and partly due to ignorance of the appropriate mathematical tools.

Since then a dramatic change has taken place and nonlinear stability has become an important branch of fluid mechanics, attracting a large number of theoreticians. It is now appreciated that the role of linear theory is to provide necessary conditions for stability, whereas for sufficiency a global theory is required. Further, iterative processes can be developed to describe the evolution of marginally stable states, according to linear theory, which lend themselves to study either by singular perturbation techniques or rigorous arguments. Again, the availability of massive computational facilities has enabled revealing studies to be made of fully nonlinear motions otherwise completely beyond the reach of present-day analysis. The range of interest in fluid mechanics has also widened enormously, convection processes in the laboratory, the ocean, the atmosphere, the interior of the earth and in astrophysics generally being intensively studied because of their importance in many phenomena of current interest. Finally, the nonlinear theory has achieved some signal successes exemplified by Stuart's calculation of the torque in rotating Couette flow and the Benjamin–Feir instability of the Stokes wave train, which give hope that it will prove to be much more than of intrinsic interest.

Now, after several years preparation, one of the most distinguished mathematicians working in this area has written a book setting out the modern position in many aspects of this challenging and difficult subject. He is obviously at home with most of the principal ideas in current use – bifurcation theory, variational principles, perturbation techniques, continuous materials – having played an important role in shaping them, and with the experimental evidence on the behaviour of fluids. The result is a book of remarkable insight, breadth and creativity which students of stability will consult and treasure for many years to come.

The first two chapters are concerned with the fundamental ideas of global stability, linear instability and bifurcation theory. An important point which needs to be understood is that linearized theory of itself cannot decide whether a particular flow is stable; for this one must consider its response to all physically acceptable disturbances. Global theory to date provides some evidence but usually it is inadequate and the next best guide is the feedback from the nonlinear terms. This leads to the notions of supercritical and subcritical bifurcations, associated with a third-order feedback, and two-sided (sometimes known as transcritical) bifurcations, which are of second order. Broadly speaking, the nonlinear theory is more successful when the bifurcation is supercritical for one can then contemplate a more or less ordered progression towards turbulence through a succession of bifurcations of increasing complexity.

The first application of these fundamental ideas is to the stability of Poiseuille flow, one of the central problems in fluid mechanics. This flow proves to be especially difficult because the linear instability, if it occurs, is overstable (i.e. an oscillatory disturbance is the first to become unstable as the Reynolds number Re is increased), the bifurcation is subcritical and the observed transition to turbulence is abrupt. The first aspect considered is that of global stability, it being established that annular flows are globally and monotonically stable if $Re < 81$. A general discussion follows on the observed flow properties which usually show a transition to turbulence at $Re \sim 2000$, unless special care is taken. Howard's method, by which for the first time upper bounds for the friction factors in turbulent flow can be stated, is reviewed next; the best value quoted exceeds the measurements by a factor of about 100. Third, the bifurcation problem is discussed, on the basis of the neutral oscillations and centred on the linear stability limit $Re = 11544$, and its subcritical nature established. This is a stimulating account of recent research and even if the experimenter finds the present state of the art to be rather depressing, the theoretician will recognize that important progress has been made.

Next Joseph turns to Couette flow between rotating cylinders, where there is more encouraging news. In particular, global stability and linear theory coincide when the two cylinders have approximately the same angular velocities, so that necessary and sufficient conditions for stability are available, and indeed linear theory agrees well with experiment over a broad range of parameter space. Also the bifurcation can be supercritical and steady, and the nonlinear theory can be in quantitative agreement with observation. This material leads naturally to the study of spiral Couette–Poiseuille flows in which rotational, sliding and pressure-gradient effects must all be considered. Fortunately, the basic flows may be written down in analytic form, considerably easing the study of their stability characteristics. The main concern is with global stability and a number of examples are treated. However, I was left with a slightly confused impression about the present state of the research and it seems that some of the most important conclusions may be found in the footnotes.

A related problem occupies chapter VII, namely the stability of flow between rotating spheres, and the material presented, both experimental and theoretical, is largely that of the author and his collaborators. The first volume of the book concludes with a series of interesting appendices on almost periodic functions, variational aspects of the theory of energy decay, a set of inequalities of use in providing bounds in the energy theory of stability, oscillation kernels in integral equations, and the stability of nearly parallel flows.

The first four chapters of the second volume of the book are primarily concerned with convective instability of motionless fluid when subject to uniform heating from below. They open with a historical review and the derivation of the fundamental equations and boundary conditions, then go on to discuss the global stability of what may be called the classical Bénard problem and its generalization to include concentration gradients. A number of interesting theorems are proved which enable a partial delineation of the region of parameter space corresponding to globally stable states to be made. Next the D'Arcy–Oberbeck–Boussinesq (DOB) equations for slow flow in a porous medium are introduced and studied at length. The special characteristics of these equations are that in the momentum equations the viscous terms are replaced by a term proportional to the velocity and the inertial terms are *virtually* neglected. When there is also internal heating the author establishes a simple criterion for two-sided bifurcation to occur; illustrations include hexagonal cells and circular containers. An instructive and penetrating discussion follows on two-sided bifurcation in Oberbeck–Boussinesq (OB) fluids, including the significance of lateral boundaries; it is likely however that some qualifications will be needed in view of recent work by Benjamin, Daniels and others.

If the fluid is not heated internally then the bifurcation is supercritical and the first-order theory for DOB fluids can be extended to include higher orders and so the stability of the convective rolls to three-dimensional disturbances may be examined. This chapter ends with a discussion of the envelope equation for the cell amplitudes in the marginally stable state using the method of multiple scales and based on ideas of Newell & Whitehead. Parallel work by Segel is ignored, which is the more curious since he was the first to recognize how lateral boundaries may be incorporated in this theory.

The variational theory of turbulence, already discussed in chapter IV in connexion with duct flows, is now applied to convection in porous materials with the aim of finding an upper bound for the heat transport. This study is of considerable interest, embracing an examination of the properties of the simplest steady convection cell at all Rayleigh numbers together with its bifurcations. Finally, a comparison with the experimental results shows remarkable agreement, although to be sure it seems that for the range of Rayleigh numbers considered the observed flows are not turbulent.

Next, a theory for viscoelastic fluids is presented which makes use of a number of the ideas already developed in this book. Much of the material is new and outside my own interests but I continue to be impressed by the experimental evidence obtained by the author about the properties of steady and unsteady motions of such fluids. The book concludes with an account of the interfacial stability theory of Dussan V.

I have already expressed my high opinion of this book as a whole and there is no question but that it may be strongly recommended to anyone who wishes to understand what has been achieved in nonlinear stability theory and to develop a firm base for future advances. There is no other book available at present which approaches it in covering so many aspects of the theory and in its understanding of the mathematical and physical processes which underlie the phenomena. Nevertheless, as I pondered over the material and considered the subject as a whole, I became aware of a feeling of dissatisfaction with the treatment of particular topics and eventually I had to qualify my admiration with some reservations.

First, the laminar boundary layer is relegated to a part of appendix E, where, although the commentary is sound and stimulating, it barely scrapes at the problem. Yet this flow is so important, and leads naturally to many other shear flows which are extensively studied, that it must be regarded as one of the central problems of stability theory attracting more interest than any other. It was disappointing that so little space could be spared, the more so because recent work, by Gaster and others, has established its value in an experimental context.

Second, on occasion I felt that the perspective taken by the author is narrow. For example, the theory of flow in porous media is relatively new to me but during my perusal of the account here, it was hard at first to convince myself that the theory was more than a mathematical toy. Only when I saw the graphs on p. 189 did I realize that there might be more to it. A study of the quoted literature soon convinced me of the physical value of the model and I learned that flows have many features to challenge the theoretician. In fact, the theory provided in the book is only part of the story and not necessarily the most interesting. Again, interfacial stability has a long history going back at least to Maxwell. Recently, important work on the shape and stability of the interface has been carried out by Finn, Padday, and others. The present discussion is significant but covers only a small fraction of the subject.

I also noticed an apparent antipathy to two useful methods in stability theory. Early on, the method of multiple scales is dismissed as fussy and made the object of mildly ribald humour. This shows a lack of historical generosity and indeed is an error of judgment for the method has been applied to the theory of instability of water waves and much else of proven worth. The author is, moreover, not averse to making use of the ideas in a roundabout way, back-tracks on his early remarks at various points and ends up by deriving the amplitude equation. Further, the theory of non-linear convection which is presented here is based on the assumptions that the lateral extent of the region if infinite and the base is uniformly heated. This is an idealized physical situation and even if we wish to apply the theory to finite domains the amplitude of the cellular motions must be allowed to vary. There is also much to be said for the view that bifurcation is a phenomenon only of such ideal situations and that any imperfections, for example conducting lateral boundaries, convert it into a smooth evolutionary process together with a possibility of non-uniqueness (and even hysteresis). The method is well suited for the generalization of the basic theory to these more general problems.

The contributions made by massive computation to our understanding of non-linear processes are largely disregarded. These include the studies initiated by Malkus and Veronis and continued by Weiss, among others, on convective processes at high Rayleigh number, which are of importance in understanding the evolution of convection cells and their relation to the emergence of turbulent convection. The numerical studies of convection in DOB fluids by Elder also merit discussion.

I suppose that the point is this: the title of the book suggests something general and rather comprehensive, whereas this is not the intention of the author. Indeed, volume II might more properly be called 'the stability of the motionless state'. The original intention was to write a monograph on global stability but this proved to be too much of a strait-jacket as the author's interest and creative power extended into so many aspects of stability theory. The scope of the book was therefore greatly widened and I surmise that it was just not possible to give an equally broad perspective to all the topics added, especially as the author was simultaneously making new contributions at a phenomenal rate. In a number of instances he seems to have concentrated on these but included just sufficient historical and peripheral material for proper understanding.

Notwithstanding these reservations, I am confident that the book is essential reading for established workers in stability, who had better keep in close touch with Joseph's work if they wish to be abreast of future developments. Beginners will benefit greatly from it, especially after having had some grounding in linearized theory. A particularly valuable and unusual service is provided for them by the author in giving a large number of examples, ranging from the easy to the difficult, which are interspersed throughout the text to enable the student to develop facility in the wide variety of ideas advocated. For all interested in the stability of fluid motions the book will be a source of pleasure and stimulation.

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